PRINCIPLES OF ANALYSIS
TOPIC V: CLUSTER POINTS AND SUBSEQUENCES

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Abstract. We define cluster points and subsequences, and prove two versions of the Bolzano-Weierstrass Theorem.

1. Cluster Points

Definition 1. Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers and let $q \in \mathbb{R}$.

We say that $(a_n)_{n=1}^{\infty}$ *clusters at* q if

$$\forall \epsilon > 0 \; \forall N \in \mathbb{N} \; \exists n \geq N \; \ni \; |s_n - C| < \epsilon.$$

In this case, we call q a *cluster point* of $(a_n)_{n=1}^{\infty}$.

Proposition 1. *Let (a_n) be a sequence in $\mathbb{R}$ which converges to $p \in \mathbb{R}$. Then p is a cluster point of (a_n).*

Proof. Let $\epsilon > 0$ and $N \in \mathbb{N}$; we wish to show that there exists $n \geq N$ such that $|a_n - p| < \epsilon$. Since (a_n) converges to p, there exists $N_0 \in \mathbb{N}$ such that $n \geq N_0$ implies $|a_n - p| < \epsilon$. Let $n = \max\{N, N_0\}$; then $n \geq N$ and $|a_n - p| < \epsilon$. □

Proposition 2. *Let (a_n) be a bounded sequence of real numbers. Then*

(a) $\limsup a_n$ *is a cluster point of* (a_n)*;*

(b) $\liminf a_n$ *is a cluster point of* (a_n)*.*

Proof. Since (a_n) is bounded, $\limsup a_n$ and $\liminf a_n$ exist as real numbers. Let $u = \limsup a_n$; we wish to show that u is a cluster point of (a_n).

Let $\epsilon > 0$ and let $N \in \mathbb{N}$; it suffices to show that there exists $m \geq N$ such that $|a_m - u| < \epsilon$. Let $u_M = \sup\{a_n \mid n \geq M\}$.

Since $u = \lim_{M\to\infty} u_M$, there exists $N_0 \in \mathbb{N}$ such that, for all $M \geq N_0$, we have $|u_M - u| < \epsilon$. Let $M = \max\{N, N_0\}$. Then $U - \epsilon < u_M < u + \epsilon$; since $u_M = \sup\{a_n \mid n \geq M\}$, there exists an element of $\{a_n \mid n \geq M\}$ between $U - \epsilon$ and u_M. Select $m \in \mathbb{N}$ with $m \geq M \geq N$ such that $u - \epsilon < a_m < u_M$. We have $u - \epsilon < a_m < u + \epsilon$, so $|a_m - u| < \epsilon$. Thus u is a cluster point of (a_n).

That $\liminf a_n$ is a cluster point can be proved similarly. □

Proposition 3. (Bolzano-Weierstrass Theorem Version I)
Every bounded sequence of real numbers has a cluster point.

Proof. The limit superior of a bounded sequence exists as a real number, and this real number is a cluster point by Proposition 2. □

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Proposition 4. *Let (a_n) be a bounded sequence in $\mathbb{R}$, and let q be a cluster point of (a_n). Then $\liminf a_n \leq q \leq \limsup a_n$.*

Proof. Suppose that $q \in \mathbb{R}$, and assume that $q > u = \limsup a_n$. Now $q - u$ is positive; let $\epsilon = \frac{q-u}{2}$. By definition of limit superior, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|\sup\{a_n \mid n \geq N\} - u| < \epsilon$. Thus for every $n \geq N$, we have $\sup\{a_n \mid n \geq N\} < u + \epsilon$, so $a_n < u + \epsilon = q - \epsilon$, and $q - a_n > \epsilon$.

This shows that q is not a cluster point; thus any cluster point must be less than or equal to $\limsup a_n$. Similarly, any cluster point must be greater than or equal to $\liminf a_n$. □

Proposition 5. *Let (a_n) be a sequence in $\mathbb{R}$. Then (a_n) converges to p if and only if p is the only cluster point of (a_n).*

Proof. We prove the double implication in each direction, using the fact that (a_n) converges (to p) if and only if $\liminf a_n = \limsup a_n$ (which equals p), as we have previously shown.

($\Rightarrow$) Suppose that (a_n) converges to p. By Proposition 1, p is a cluster point of (a_n), and we wish to show it is the only cluster point. Let q be a cluster point; we wish to show that $q = p$.

By Proposition 4, we have $\liminf a_n \leq q \leq \limsup a_n$. Because (a_n) converges to p, we know that $\liminf a_n = \limsup a_n = p$. Thus $q = p$.

($\Leftarrow$) Suppose that p is the only cluster point of (a_n). Then $\liminf a_n = p = \limsup a_n$. This shows that (a_n) converges to p. □

2. Subsequences

Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence of real numbers. A *subsequence* of a is the composition $a \circ n$ of a with a strictly increasing sequence $n : \mathbb{N} \to \mathbb{N}$ of positive integers.

If we denote the sequence a by $(a_n)_{n=1}^{\infty}$ and the sequence n by $(n_k)_{k=1}^{\infty}$, then we denote the subsequence by $(a_{n_k})_{k=1}^{\infty}$.

Proposition 6. *Let $n : \mathbb{N} \to \mathbb{N}$ such that $n \mapsto n_k$ be an increasing sequence. Then $n_k \geq k$.*

Proof. By induction on k.

For $k = 1$, we have $n_k = n_1 \geq 1$, since $n_k \in \mathbb{N}$.

Assume that $n_k \geq k$; then $n_k + 1 \geq k + 1$. Since n is increasing, $n_{k+1} > n_k$, so $n_{k+1} \geq n_k + 1$. Thus $n_{k+1} \geq n_k + 1 \geq k + 1$. □

Proposition 7. *Let (a_n) be a sequence of real numbers and let $p \in \mathbb{R}$. Then (a_n) converges to p if and only if every subsequence of (a_n) converges to p.*

Proof. We prove both directions.

($\Leftarrow$) Note that a sequence is a subsequence of itself. Thus if every subsequence of (a_n) converges to p, then in particular the sequence itself converges to p.

($\Rightarrow$) Suppose that $\lim a_n = p$. Let (a_{n_k}) be a subsequence of (a_n), and let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|a_n - p| < \epsilon$. Thus for $k \geq N$, we have $n_k \geq K \geq N$, so $|a_{n_k} - p| < \epsilon$. □

Proposition 8. *Let (a_n) be a sequence of real numbers. Then (a_n) has a monotonic subsequence.*

Proof. This proof follows Ross, which in turn follows D. J. Newman's *A Problem Seminar.*

Let's say that the i^{th} term of (a_n) is *dominant* if $a_j < a_i$ for every $j > i$.

Case 1: There are infinitely many dominant terms. In this case, set

$$n_1 = \min\{n \in \mathbb{N} \mid a_n \text{ is dominant}\}.$$

Then recursively set

$$n_{k+1} = \min\{n \in \mathbb{N} \mid a_n \text{ is dominant and } n > n_k\};$$

this set is nonempty by the hypothesis of this case. Then (a_{n_k}) is a decreasing sequence.

Case 2: There are finitely many dominant terms. In this case, set

$$n_0 = \max\{n \in \mathbb{N} \mid a_n \text{ is dominant}\}.$$

Then recursively set

$$n_{k+1} = \min\{n \in \mathbb{N} \mid a_n > a_{n_k} \text{ and } n > n_k\};$$

this set is nonempty because a_{n_0} was the last dominant term. Now (a_{n_k}) is an increasing sequence. □

Proposition 9. (Bolzano-Weierstrass Theorem Version II)
Every bounded sequence of real numbers has a convergent subsequence.

Proof. It is clear that if a sequence is bounded, then every subsequence is also bounded. Thus a bounded sequence has a bounded monotonic subsequence, which must converge. □

3. Subsequential Limits

Definition 2. We say that q is a *subsequential limit* of (a_n) if there exists a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} a_{n_k} = q$.

Proposition 10. *Let (a_n) be a sequence of real numbers, and let $q \in \mathbb{R}$. Then q is a cluster point of (a_n)if and only if q is a subsequential limit of (a_n).*

Proof. Suppose that q is a cluster point. Then for every $N \in \mathbb{N}$ there exists $n \geq N$ such that $|a_n - q| < \frac{1}{N}$.

Set

$$n_1 = \min\{n \in \mathbb{N} \mid |a_n - q| < 1\},$$

and inductively set

$$n_{k+1} = \min\{n \in \mathbb{N} \mid |a_n - q| < \frac{1}{n} \text{ and } n > n_k\}.$$

That these sets are nonempty is assured by the fact that (a_n) clusters at q. Then (a_{n_k}) is a subsequence of (a_n) which converges to q.

Suppose that (a_{n_k}) is a subsequence which converges to q. Let $\epsilon > 0$ and let $N \in \mathbb{N}$. Let K be so large that $k \geq K \Rightarrow |a_{n_k} - q| < \epsilon$. Let $n = \max\{N, K\}$. Then $n \geq N$, so $n_k \geq N$. Moreover, $n \geq K$, so $n_k \geq K$ and $|a_{n_k} - q| < \epsilon$. □

Remark 1. We have previously seen that every bounded sequence has a cluster point, and we have just seen that every cluster point is the limit of a subsequence. This produces an alternate proof of the Bolzano-Weierstrass Theorem Version II.

Proposition 11. *Let (a_n) be a bounded sequence in $\mathbb{R}$.*
Then there exist monotonic subsequences of (a_n) which converge to $\limsup a_n$ and $\liminf a_n$.

Proof. We have seen that $\limsup a_n$ and $\liminf a_n$ are cluster points, and that cluster points are subsequential limits. Since every sequence has a monotonic subsequence, the result follows. □

4. Problems

Problem 1. Construct a divergent sequence (a_n) of real numbers such that (a_{mk}) converges for every $m \in \mathbb{N}$, $m \geq 2$.

Solution. We use the fact that there are infinitely prime numbers.

Define

$$a_n = \begin{cases} 1 & \text{if } n \text{ is prime ;} \\ 0 & \text{otherwise .} \end{cases}$$

Since there are infinitely many primes, $\limsup a_n = 1$. Since there are infinitely many nonprimes, $\liminf a_n = 0$. Thus (a_n) does not converge.

However, for any $m \in \mathbb{N}$ with $m \geq 2$, mk is not prime for $k \geq 2$, so $a_{mk} = 0$ for all $k \geq 2$. Thus $\lim_{k\to\infty} a_{mk} = 0$, and (a_{mk}) converges. □

Problem 2. Let (a_n) and (b_n) be bounded sequences of positive real numbers, and suppose that $0 \in \mathbb{R}$ is a cluster point of the sequence $(a_n b_n)$. Show that 0 is a cluster point of either (a_n) or of (b_n).

Problem 3. Let (a_n) and (b_n) be bounded sequences of positive real numbers, and suppose that $c \in \mathbb{R}$ is a cluster point of the sequence $(a_n + b_n)$. Show that there exist cluster points a of (a_n) and b of (b_n) such that $c = a + b$.

Problem 4. Let (a_n) and (b_n) be bounded sequences of positive real numbers, and suppose that $c \in \mathbb{R}$ is a cluster point of the sequence $(a_n b_n)$. Show that there exist cluster points a of (a_n) and b of (b_n) such that $c = ab$.

Problem 5. Construct sequences (a_n) and (b_n) such that a is a cluster point of (a_n) and b is a cluster point of (b_n), but $a + b$ is not a cluster point of $(a_n + b_n)$.

Problem 6. Construct sequences (a_n) and (b_n) such that a is a cluster point of (a_n) and b is a cluster point of (b_n), but ab is not a cluster point of $(a_n b_n)$.

Problem 7. Let (a_n) and (b_n) be bounded sequences of positive real numbers, and suppose that $(a_n b_n)$ has a subsequence which converges to 0. Show that either (a_n) or (b_n) has a subsequence that converges to 0.

Problem 8. Let (a_n) and (b_n) be bounded sequences of positive real numbers, and suppose that $(a_n + b_n)$ has a subsequence which converges to $c \in \mathbb{R}$. Show that there exists a subsequence of (a_n) which converges to $a \in \mathbb{R}$ and a subsequence of (b_n) which converges to $b \in \mathbb{R}$ such that $c = a + b$.

Problem 9. Let (a_n) and (b_n) be bounded sequences of positive real numbers, and suppose that $(a_n b_n)$ has a subsequence which converges to $c \in \mathbb{R}$. Show that there exists a subsequence of (a_n) which converges to $a \in \mathbb{R}$ and a subsequence of (b_n) which converges to $b \in \mathbb{R}$ such that $c = ab$.

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